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A discrete map for the dripping faucet dynamics

L. Renna *

Dipartimento di Fisica dell'Università and Istituto Nazionale di Fisica Nucleare, via per Arnesano, 73100 Lecce, Italy

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Abstract

A discrete map for the time intervals between drops falling off a leaky tap is presented. The map reproduces the dynamics observed in real systems. Attractors and bifurcation diagrams similar to experimental ones are shown and discussed. © 1999 Elsevier Science B.V. All rights reserved.

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The behavior of dripping liquid from a faucet has been experimentally investigated by several authors [1–11]. A wide class of phenomena has been reported, including chaos, thus the dripping faucet can be considered a sort of 'model system' of complex behavior. In the experiments the time intervals between successive liquid drop detachments at different values of flow rate are measured. The data are displayed plotting time-delay diagrams and dripping spectra (bifurcation diagrams with flow rate as control parameter). Theoretical studies [12–17] are essentially based on the variable-mass oscillator of Shaw [1], where a drop hanging from a nozzle is described as a variable mass attached to a spring and subjected to the gravitational force and to a friction force. Substantial improvements over this model were presented in previous papers [14] and [15], where the importance of the discontinuity at the critical point (during the breaking-off) is emphasized [18]. Retaining this suggestion and considering the behavior of the forming drop in real systems [8], we propose in this paper a discrete map [19], which considerably simplifies the mapping technique of Ref. [16], where a numerical solution of a nonlinear equation is required. The discrete map reproduces the features detected experimentally for a leaky tap, such as, for example, periodicity, period-doubling, multiperiodicity, quasiperiodicity, pitchfork and tangent bifurcation, Hopf bifurcation, inverse cascade, coexisting attractors, strange attractors, and so on. The considerations that are at the basis of model are also discussed, together with the possible improvements. As far as we know, so far no physically based maps for the dripping faucet have appeared in the literature.

* E-mail: renna@le.infn.it

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In Ref. [16] the centre of mass of a drop hanging from an orifice is approximately described by the equation

$$X(T) = [A \sin \Omega(T) T + B \cos \Omega(T) T] \mathrm{e}^{-T/M(T)} + M(T)/K, \tag{1}$$

where

$$\Omega^2(T) = K/M(T), \tag{2}$$

$X = x/x_c$ is the displacement normalized to the critical displacement x_c, T is the time normalized to $\tau = (x_c/g)^{1/2}$, g is the acceleration of gravity, M is the mass normalized to $\mu = b\tau$, b is the coefficient of friction, K is a spring constant normalized to b/τ and F represents the flow rate normalized to b.

The mass is supposed to vary linearly with the time

$$M(T) = m + FT, \tag{3}$$

where m represents the mass of the residue after the falling of the previous drop.

The breaking away of the drop is simulated by reducing the mass, at the critical point $X_c = 1$, by a quantity proportional to its momentum

$$\Delta M = \alpha M V, \tag{4}$$

where α is a parameter. The condition (4) was first used in Refs. [12,13] and independently in Ref. [14].

Experimentally, at threshold, the forming drop of mass M undergoes a stretching forming a neck [5,6,8], so that it is reasonable to suppose that, after the fall-off of the drop, the residual restarts with velocity V_c at the point

$$X_0 = 1 - R \frac{\Delta M}{M}, \tag{5}$$

where

$$R = \left(\frac{3 \Delta M}{4 \pi D} \right)^{1/3} \tag{6}$$

represents the drop radius and D the liquid density normalized to μ/x_c^3.

Thus a mapping, that is the series of time intervals between each drop, can be obtained by solving numerically the equation

$$X(T) = 1. \tag{7}$$

Eqs. (5) and (6) are the basis of the 'spherical drop model' proposed in Ref. [14].

Eq. (1) approximates well enough the behavior of the real drop in formation, as one can see by comparing the plot in Fig. 1 (continuous curve) with those reported in Fig. 7 of Ref. [8], where it is shown that the centre of gravity of the fluid oscillates like a damped oscillator as it reaches the critical point, then a break-off of a drop occurs. We emphasize that the nonlinearity required to yield chaos is the sudden change at the threshold. Therefore it is not essential that $X(T)$ follows the actual path of the drop centre of mass, but only that it reproduces dynamical effects around the critical point [16,17]. In Fig. 1, the broken line represents the last term of Eq. (1) that typically follows the mean pattern of $X(T)$ [as given by Eq. (1)]. Thus, in order to obtain a crude approximation of drop displacement, one could put at the critical point

$$X(T) \approx \frac{m + FT}{K} = 1, \tag{8}$$

and reversing it we obtain $T_c = (K - m)/F$. If Eq. (8) is used together with the derivative of Eq. (1) at X_c one gets bifurcation diagrams which are typical

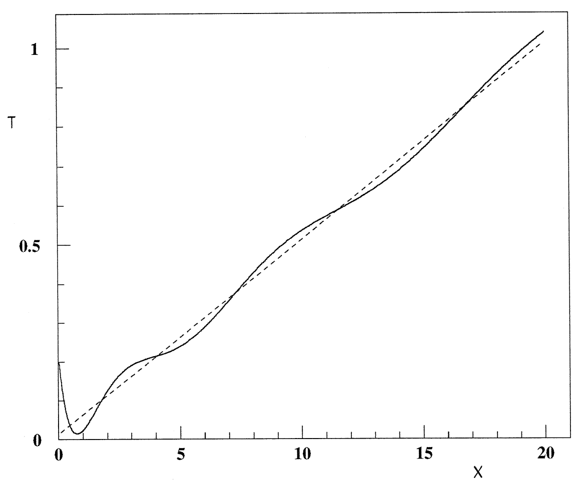

Fig. 1. Plot of the position $X(T)$ (continuous curve) and of its second term (broken line) in Eq. (1).

of complex behavior, with attractors showing a smaller dimensionality (as this approximation corresponds to a very large coefficient of dissipation). This is caused also by the omission of the dependence of $X(T)$ on $V_c = V_0$, which represents both the velocity of the drop at the critical point and the velocity of the residue at X_0. In fact the dripping faucet behaves as a sort of relaxation oscillator, where self-stimulated oscillations originate between successive drops. Therefore it is essential, in order to recover dynamics, that $X(T)$ be also a function of V_0. Insertion of V_0 accounts for the difference of the displacement from its mean value as given by Eq. (8). It seems reasonable to assume an inverse dependence between the drop formation time and its initial velocity; thus we can write

$$T_c \approx \frac{K-m}{F+V_0}. \tag{9}$$

This approximation is equivalent to adding to Eq. (8) the term $V_0 T/K$, which changes the slope of the straight line representing the mean displacement of each drop, as occurs in real experiments [8]. Adding this corrective term to Eq. (8) one obtains, at the critical point, an approximated expression for the displacement $X(T)$ that can be reversed in order to obtain the dripping time T_c. Thus Eq. (9) gives a

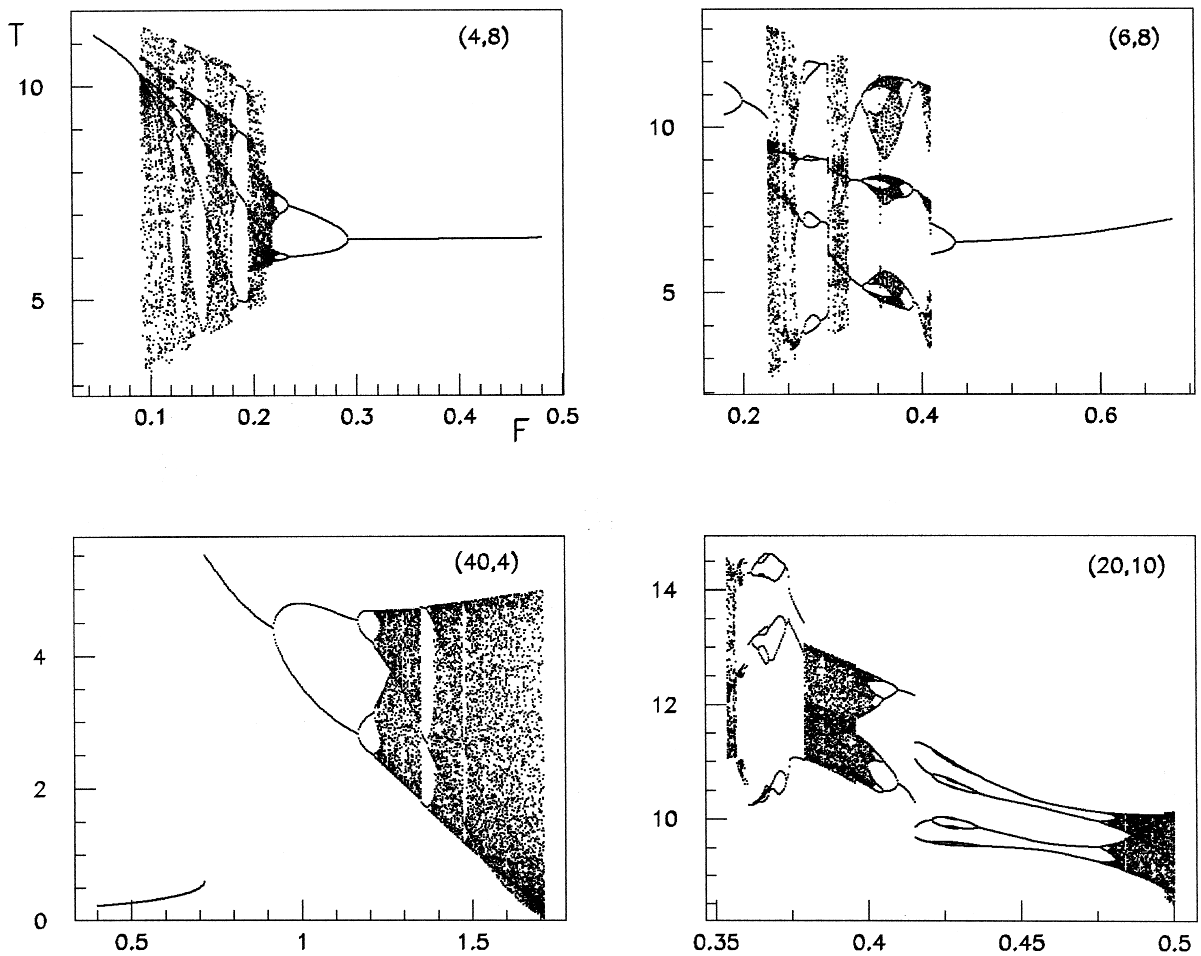

Fig. 2. Bifurcation diagrams plotted against flow rate F. On each frame the values of the pair (K,α) are indicated. 50 points (after a transient of 1000 drops) are used at each value of F, and 25×10^3 points for one whole plot.

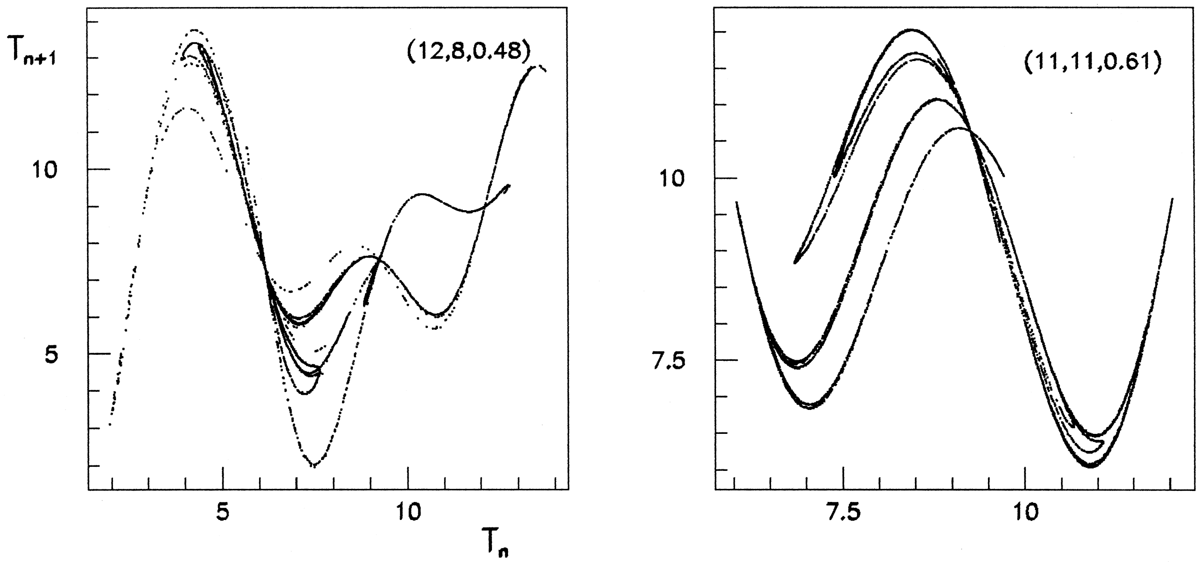

Fig. 3. Return maps T_{n+1} versus T_n. On each plot the values of (K,α,F) are inserted; 10^4 points are used (after a transient of 10^3). A similarity of qualitative behavior with some experimental attractors can be observed, see Fig. 7(f) and 11(d) of Ref. [3].

convenient valuation of the formation time of a drop with staring mass m and speed V_0.

The map can be built as follows. Changing a little the notations, we call m_n, υ_n and x_n the initial mass, speed and position of a drop n, then T_n as given by (9) represents the time interval between the actual drop detachment and the previous one. Let V_n be the time derivative of Eq. (1) calculated at T_n, and M_n the mass of the drop at the critical point, then

$$T_n = \frac{K - m_n}{F + \upsilon_n}; \quad V_n = V(m_n, \upsilon_n, x_n, T_n);$$
$$M_n = m_n + FT_n. \tag{10}$$

Eq. (10) express the relations, for each forming drop, between the time T_n, the speed V_n and the mass M_n at the critical point $(X_n = 1)$ as function of the corresponding starting quantities $t_n = 0$ (not marked), υ_n, m_n and x_n.

By using Eqs. (4) and (5) the following map is obtained

$$x_{n+1} = 1 - \left(\frac{3\alpha M_n V_n}{4\pi D} \right)^{1/3}, \quad m_{n+1} = M_n(1 - \alpha V_n),$$
$$\upsilon_{n+1} = V_n, \tag{11}$$

where m_{n+1}, υ_{n+1} and x_{n+1} represent the mass, speed and position of the residue after the fall-off of the drop n, namely the corresponding initial conditions of the drop $(n+1)$.

Simulations are performed using T_n [Eq. (10)], which is the quantity measured in experiments.

In Fig. 2 typical bifurcation diagrams obtained with the map (11) show complex dynamical behavior as the control parameter (flux) is varied. The plots correspond to different values of the parameters (K,α) with the value of the parameter D kept fixed at 1. [1] Periodic behavior, period-doubling, crisis and chaos are evidenced in the diagrams. We can observe some difference between these dripping spectra. At high values of K multiperiodic behavior extends over wide regions of spectrum and period doubling

[1] Moreover, for each flow rate F, return maps are always calculated using the same initial conditions $(x_0 = 0,\ \upsilon_0 = 0.001,\ m_0 = 0.01$, if not differently indicated). The reason is that the model can produce coexistence of attractors. Bifurcation diagrams are instead calculated by retaining, for the first drop at a given F, the values calculated from the last drop of the previous F: this is reminiscent of experimental diagrams obtained by emptying a large reservoir.

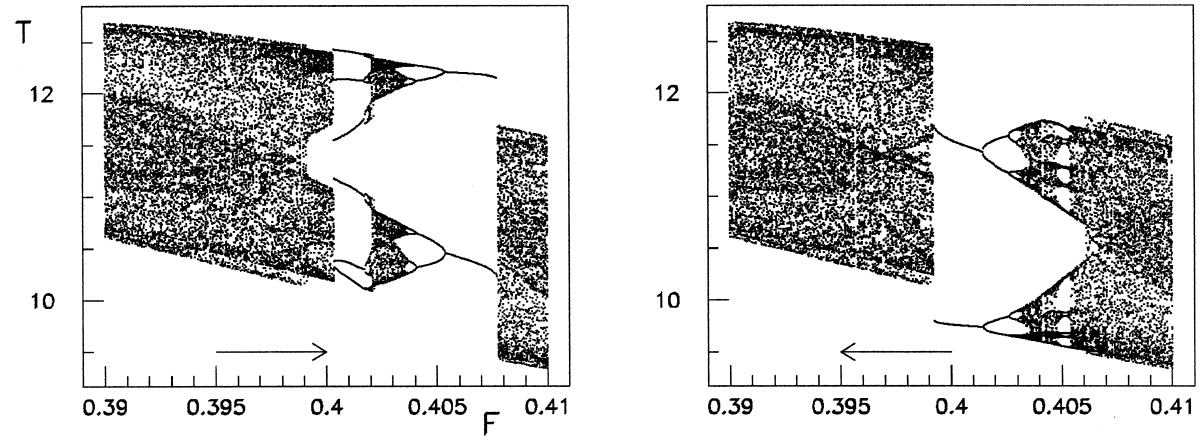

Fig. 4. Bifurcation diagrams showing hysteresis ($K = 30$, $\alpha = 10$). (a) F is increased; (b) F is decreased.

dominates; at low values of K inverse cascade is observed. High values of α increase the variety of transitions and contract the region of flow rate after which the dripping transforms into continuous flow.

Besides exhibiting transitions typical of the real dripping faucet, the model also yields strange attractors similar to the experimental ones. In Fig. 3 examples of return maps are plotted which qualitatively reproduce behaviors found in the real system, such as those reported in Figs.7 and 11 of Ref. [3].

Hysteresis is presented in Fig. 4 where different behaviors are observed depending on whether F is increased or decreased. This feature has been found in the experimental systems. The variable-mass oscillator shows hysteresis thanks to insertion of a rebound for the residue at the breaking-off point, as that represented by Eq. (5).

A transition from period-1 behavior to chaos through tangent intermittence is illustrated in Fig. 5(a) by means of a time series representation around

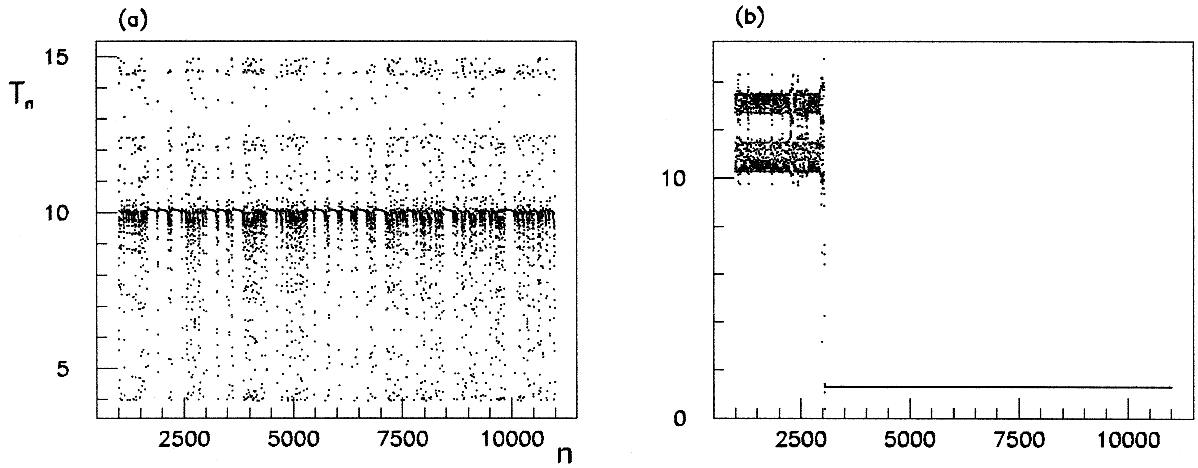

Fig. 5. (a) Tangent intermittence between chaos and period-1 attractors ($K = 9$, $\alpha = 9$, $F = 0.3778$). (b) The sequence of drip intervals shows a sudden change from two chaotic bands to periodic regime ($K = 4$, $\alpha = 10$, $F = 0.01$). Both behaviors have been found in the experimental time series [6].

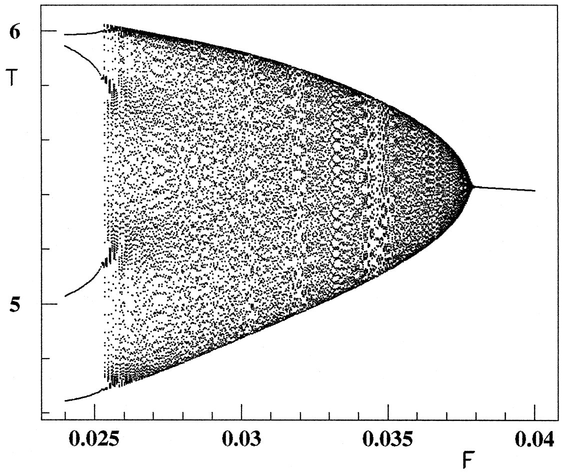

Fig. 6. Dripping spectrum showing an Hopf bifurcation.

the bifurcation point. An uncommon feature of instability is shown in the time series of Fig. 5(b), where two chaotic intermittent bands abruptly regularize on a period-1 behavior. A drastic change in the mean drop frequency is observed: analog behavior has been visualized for a real dripping faucet (see Fig. 8(b) of Ref. [6]).

Closed loops attractors are found at low values of (K, α). Inspection of the corresponding spectra shows transitions such as that reported in Fig. 6, which is characterized by an evolution from a period-4 to a period-1 frequency through a quasiperiodic development (an Hopf bifurcation). For $K = 4$ and $\alpha = 4$ an evolution from a period-5 to a period-1 frequency happens, similar to that reported in Ref. [7] for the real leaky tap.

The preliminary results presented show that the map (11) produces a large class of phenomena which reproduce most of the dynamics of a dripping faucet. The use of a map reduces the computational time, whereas the physical meaning of parameters remains nearly unchanged. Moreover, the method presented here gives concrete and useful suggestions for building, in the spirit of mass-on-a-spring model, a discrete map convenient for a close description of behavior of actual leaky tap. On this subject some warnings should be made. First, the link between model and physical parameters is not precisely established; it can be, for example, that α or K have some dependence on the flow rate. Preliminary studies [20] and results from fluid dynamical calculations [21] show indications favorable to this hypothesis. Second, the physical mechanism which simulates the detachment of the drop, modeled by Eqs. (4) and (5), is crucial in order to yield chaos: indeed, it produces the appropriate correlations between successive drops, but unlike conditions at the threshold can give different results [14,15]; still it seems clear that the rebound condition does depend somehow on the mass of the falling drop. Finally, the approximation for the position of growing mass at the critical point which allows one to get a formula for the drop formation time [here Eq. (9)] demands further analyses. Studies are in progress in order to investigate these points.

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References

[1] P. Martien, S.C. Pope, P.L. Scott, R.S. Shaw, Phys.Lett. A 110 (1985) 399.
[2] H.N. Núñez Yépez, A.L. Salas Brito, C.A. Vargas, L.A. Vicente, Eur. J. Phys. 10 (1989) 99.
[3] X. Wu, Z.A. Schelly, Physica D 40 (1989) 433.
[4] R.F. Cahalan, H. Leidecker, G.D. Calahan, Comput. Phys. 4 (1990) 368.
[5] K. Dreyer, F.R. Hickey, Am. J. Phys. 59 (1991) 619.
[6] J.C. Sartorelli, W.M. Gonçalves, R.D. Pinto, Phys. Rev. E 49 (1994) 3963.
[7] R.D. Pinto, W.M. Gonçalves, J.C. Sartorelli, M.J. de Oliveira, Phys. Rev. E 52 (1995) 689.
[8] M.S.F. da Rocha, J.C. Sartorelli, W.M. Gonçalves, R.D. Pinto, Phys. Rev. E 54 (1996) 2378.
[9] J.G. Marques da Silva, J.C. Sartorelli, W.M. Gonçalves, R.D. Pinto, Phys. Lett. A 226 (1997) 269.
[10] W.M. Gonçalves, R.D. Pinto, J.C. Sartorelli, M.J. de Oliveira, Physica A 257 (1998) 385.
[11] R.D. Pinto, W.M. Gonçalves, J.C. Sartorelli, I.L. Caldas, M.S. Baptista, Phys. Rev. E 58 (1998) 4009.
[12] G.I. Sánchez Ortiz, A.L. Salas Brito, Phys. Lett. A 203 (1995) 300.
[13] G.I.Sánchez Ortiz, A.L. Salas Brito, Physica D 892 (1995) 151.
[14] A. D’Innocenzo, L. Renna, Int. J. Theor. Phys. 35 (1996) 941.

[15] A. D'Innocenzo, L. Renna, Phys. Rev. E 55 (1997) 6776.
[16] A. D'Innocenzo, L. Renna, Phys. Lett. A 220 (1996) 75.
[17] A. D'Innocenzo, L. Renna, Phys. Rev. E 58 (1998) 6847.
[18] P.A. Bernhardt, Physica D 52 (1991) 489.
[19] S.Greco, Thesis, Dipartimento di Fisica, Università di Lecce (1999) (unpublished)
[20] L. Renna, ''The mass on spring model for the dripping faucet – An improvement'', in: M. Boiti, L. Martina, F. Pempinelli, B. Prinari, G. Soliani (Eds.), Proc. Workshop Nonlinearity, Integrability and all that twenty years after NEEDS'79, Scientific World, Singapore, 1999.
[21] N. Fuchikami, S. Ishioka, K. Kiyono, LANL e-print ''Simulation of a Dripping Faucet'' (chao-dyn/9811020, 1998); K. Kiyono, N. Fuchikami, lanl archive preprint ''Dripping Faucet Dynamics Clarified by an Improved Mass-Spring Model'' (chao-dyn/9904012, 1999)