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Perfect and imperfect pitchfork bifurcations in a V-shaped spring-mass system: Comment on "Hysteresis in a simple V-shaped spring-mass system" [Am. J. Phys. 89, 663–665 (2021)]

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In a recent article,[1] Ong described how saddle-node bifurcations arise in a simple system comprised of two identical springs connected in a symmetrical V-shaped configuration to a mass m, such as that depicted in Fig. 1. Readers may be interested to note, however, that the same spring-mass system can also be used to study both perfect and imperfect pitchfork bifurcations.[2,3]

To see how this works, recall that the equilibrium positions y in Ong's system are given by solutions to[1]

$$f - 2ky\left[1 - \frac{l_0}{\sqrt{x^2 + y^2}}\right] = 0, \tag{1}$$

where y is the vertical equilibrium position of the mass, f is a forcing parameter (which includes gravity), $2x$ is the horizontal separation between the points securing the springs, k is the spring constant, and l_0 is each spring's natural length (see Fig. 1). Ong studied *saddle-node bifurcations* by exploring how the number of possible equilibrium configurations changes if the forcing f is varied, with x, k, and l_0 held fixed. Here, however, we study *pitchfork bifurcations* by exploring how the number of possible equilibrium configurations changes if the separation x is varied instead, with f, k, and l_0 held fixed. As we shall see, pitchfork bifurcations arise in two different ways depending on the value of the forcing f.

If $f = 0$, then Eq. (1) may be solved to give either

$$y = 0 \quad \text{or} \quad y = \pm\sqrt{l_0^2 - x^2}, \tag{2}$$

where the latter solutions are possible only if

$$x \leq x_c, \quad \text{where} \quad x_c = l_0. \tag{3}$$

This change in the number of possible equilibria when $x = x_c$ is called a *perfect pitchfork bifurcation*, as shown schematically in Fig. 2. Notice that if $x > l_0$, then the springs are under tension, such that only one equilibrium is possible, with the mass at the centre $y = 0$. If $x < l_0$, however, then three equilibria are permitted: two stable, symmetric equilibria $y = \pm[l_0^2 - x^2]^{1/2}$, with the springs relaxed at the natural length $l_0 = \sqrt{x^2 + y^2}$; and one unstable equilibrium $y = 0$, with the springs pushing against each other. Observe from Fig. 2 that transitions between the stable equilibria are continuous at $x = x_c$.

If $f > 0$, then it is not easy to solve Eq. (1) for y, and the situation becomes more complicated. One can, however, demonstrate that the number of *possible* equilibria changes from three to one (or vice versa) whenever x passes through the critical value[3]

$$x = x_c, \quad \text{where} \quad x_c(f) = l_0\left[1 - (f/2kl_0)^{2/3}\right]^{3/2}. \tag{4}$$

In this case, the change in the number of possible equilibrium configurations when $x = x_c$ is called an *imperfect pitchfork bifurcation* (see Fig. 3).

It may be shown for the imperfect pitchfork bifurcation that increasing x through x_c (e.g., by pulling the points of attachment apart from one-another) can lead to the system shifting from a stable equilibrium configuration $y < -y_c < 0$ (with the mass above the springs) to the one with $0 < y \leq y_s$ (mass below the springs), where[3]

$$y_c = \left[l_0^{2/3} x_c^{4/3} - x_c^2\right]^{1/2} \tag{5}$$

and

$$y_s = y_c\left[(l_0/x_c)^{2/3} + (1 + y_c^2 l_0^{2/3}/x_c^{8/3})^{1/2}\right]. \tag{6}$$

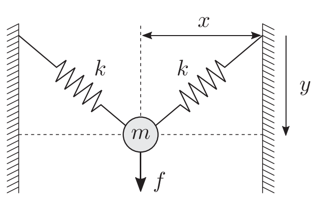

Fig. 1. Symmetric V-shaped spring-mass system (Ref. 1); it is assumed that the mass m is constrained to move in the y-direction only (e.g., by sliding on a smooth vertical wire). The y-axis is positive in the "downward" direction.

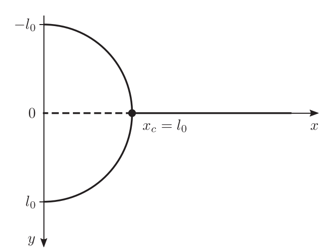

Fig. 2. Possible equilibrium positions y as a function of x when $f = 0$ (solid lines depict stable equilibria, dashed lines unstable) (Ref. 3), with the y-axis positive in the "downward" direction (as in Fig. 1). The springs are relaxed at the natural length l_0 everywhere on the semi-circle defined by $x^2 + y^2 = l_0^2$. The perfect pitchfork bifurcation occurs at $x = x_c = l_0$.

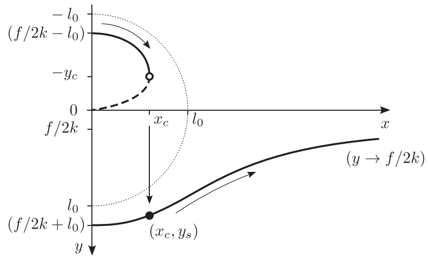

Fig. 3. Possible equilibrium positions y as a function of x for a fixed value of $f > 0$ (solid lines depict stable equilibria, dashed lines unstable) (Ref. 3), with the y-axis positive in the 'downward' direction. The imperfect pitchfork bifurcation occurs when $x = x_c(f)$; thus, if the system is initially in a stable equilibrium with $y < 0$, then increasing x through x_c will result in the equilibrium shifting discontinuously from $y = -y_c$ to $y = y_s$ (arrows). Crossing the semi-circle $x^2 + y^2 = l_0^2$ (dotted line) corresponds to the springs changing from a compressed state ($\sqrt{x^2 + y^2} < l_0$) to an extended state ($\sqrt{x^2 + y^2} > l_0$).

Such transitions between equilibria take the springs from a compressed state (with the downward force supported by the springs' reaction) to an extended state (with the downward force supported by the springs' tension) and are, therefore, discontinuous at $x = x_c$ (see Fig. 3). Note that once the system has shifted from $-y_c$ to y_s, decreasing x through x_c (e.g., by pushing the points of attachment closer together) will simply lead to the mass sinking further on the $y > 0$ equilibrium; it will not make the mass "jump" back to the $y < -y_c < 0$ configuration. In this respect, the transition at x_c is "non-reversible."

Ultimately, it may be shown that the saddle-node bifurcations discussed by Ong,[1] and the pitchfork bifurcations described above, are both features of a *cusp catastrophe*,[2,3] meaning that the system exhibits a wider range of threshold phenomena than it has been possible to summarise in our *Comment* here (see, e.g., our supplementary analysis published in another context).[3] Given such rich behaviour, therefore, the simplicity of Ong's system makes it an ideal candidate for developing undergraduate practical work on non-linear effects more generally. Indeed, the presence of bistability suggests that if friction is considered, then driving the system may even lead to chaotic behaviour analogous to that exhibited by a Duffing oscillator.[4] We look forward to investigate such possibilities in future publications.

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