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On unsteady rolling motion of spheres in inclined tubes filled with incompressible Newtonian fluids

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ABSTRACT

The characterization of the motion of immersed bodies in fluids is existent in several manufacturing operations. In the current study, the unsteady motion of a spherical particle rolling down an inclined tube submerged in an incompressible Newtonian environment has been studied using experimental data of Chhabra et al., for a wide range of Reynolds numbers. An analytical expression for instantaneous particle velocity was gained using a non-perturbation series based technique so-called homotopy perturbation method first proposed by J.H. He. Therefore, acceleration and position of the particle are derived simply. Equation of motion was solved generally and for some realistic combinations of solid–liquid. The rolling behavior of the particle was studied clearly with the analytical method and had shown for different physical conditions. The effects of particle diameter (size), continues phase viscosity and inclination angle was studied. As an important result it was found that the inclination angle does not affect the acceleration duration. Present research shows the capability and effectiveness of HPM and exhibits a new application of this method for non-linear problems. Moreover, the limited studies for particles motion in contact with a solid boundary are extended.

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1. Introduction

Sedimentation is a phenomenon of both basic and practical importance. It is extensively adopted to estimate or measure the physical properties of an entity, or to separate entities. Usual examples contain hydraulic transfer slurry systems for coal and ore transportation, thickeners, mineral processing, solid–liquid mixing, fluidization equipment, drilling for oil and gas, geothermal drilling. In all of these processes, drag coefficients and terminal velocities are important design parameters. A large number of correlations have been developed and presented in the literature relating the drag coefficient, C_D , to the Reynolds number, Re , for particles of spherical shape falling at their terminal velocities. The settling of an entity, including a solid particle, bubble, or drop, both in a Newtonian fluid and in a non-Newtonian fluid, was reviewed by Clift et al. [1], Khan and Richardson [2], Haider and Levenspiel [3] and Chhabra [4]. It was pointed out that if the Reynolds number is adequately high, wakes are formed in the rear part of the particle, and the Stokes law becomes inappropriate.

Although available results for the sedimentation of a spherical particle in an unbounded medium are sufficient in the technical literature, those for the case when a solid boundary is present

and the particle comes in contact with a solid boundary which is ubiquitous in nature and industry, are fairly limited. There are so many situations in industry and nature in which particles are in contact with a wall. In these conditions, a particle experiences rolling (with or without sliding) motion in the surrounding fluid. Typical examples include the flow in the rolling ball viscometer which involves the measurement of the rolling velocity of a tightly fitting sphere in an inclined tube and debris or sediment transport flow in canal or river beds in coastal areas. Another very practical situation in which we need to know about the rolling particles motion is related to silos and pipes: the hydraulic transporting of coarse solid particles in inclined pipes also includes rolling of particles along the wall of the pipe especially when the pipe–line works in the moving bed region. The presence of a boundary has the effect of slowing down the sedimentation of a particle. Many observations show that the hindered falling velocity is proportional to the free (terminal) falling velocity of the solid particles. For this reason, there has been a great interest in predicting the free falling velocity of solid particles in liquids. The normal procedure for estimating the free falling velocity is through the use of the drag coefficient, C_D , which is defined as the ratio of the force exerted on the falling particle due to viscous drag to the product of a characteristic area of the particle and the kinetic energy per unit volume. Unfortunately, there are few studies in the literature in the filled of rolling particles and the major part of the available investigations are

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Nomenclature

A	general differential operator	Re	Reynolds number
ac	acceleration (m/s^2)	r	independent variable
B	boundary operator	t	time (s)
C_D	drag coefficient	u	velocity (m/s)
D	pipe diameter (m)	α	inclination angle ($^\circ$)
d_p	particle diameter (m)	v	auxiliary variable
$f(r)$	given analytical function	μ_c	continues dynamic viscosity (kg/ms)
g	acceleration due to gravity (m/s^2)	ρ_c	continues phase density (kg/m^3)
H	homotopy	ρ_p	particle density (kg/m^3)
m	particle mass (kg)	Γ	boundary of domain
N	non-linear part of equation	Ω	domain
L	linear part of equation		

related to the use of a rolling ball viscometer to measure viscosity of liquids [5,6]. Hasan [7] studied the role of wall effect on the rolling velocity of spherical particles in Newtonian media. He found a very limited correlation for $(d/D) > 0.707$ as follows:

$$C_D = \frac{15.717}{Re} \left(1 - \left(\frac{d}{D} \right) \right)^{-2.5} \quad (1)$$

where Re , d , and D , are the particle Reynolds number, particle diameter and tube inner diameter respectively. Chhabra et al. [8] presented a valuable experimental work for drag on spheres in rolling motion in inclined smooth tubes. They used an enough number of sphere made of glass and steel with four smooth wallled glass tubes of different diameter. They used numerous aqueous solutions of glycerol and glucose syrup to cover a wide range of Reynolds number. The angles of inclination and sphere-to-tube diameter ratios were varied from 3° to 30° and 0.114 to 0.58, respectively. Therefore, the Reynolds number range was $10^{-6} < Re < 3000$. They had 900 data points to define their empirical correlations. It was concluded that the sphere-to-wall diameter ratio (d/D) , is not a significant parameter at the 95% confidence level. Consequently, the authors presented a three-part equation as follows:

$$C_D = \frac{225}{Re}, \quad Re < 1 \quad (2-a)$$

$$C_D = 1 + \frac{235}{Re}, \quad 1 \leq Re \leq 250 \quad (2-b)$$

$$C_D = 1.23 + \frac{177.5}{Re}, \quad 250 < Re \quad (2-c)$$

Eq. (2) predicts the transition points within about 5%, and correlates the experimental set of data within an average error of 8%. To describe a general correlation covering the experimental data we describe a new correlation using Chhabra et al. experimental points [8] as:

$$C_D = 1.2 + \frac{190}{Re} + \frac{1.003 \times 10^{-7}}{Re^2}, \quad 10^{-6} \leq Re \leq 3000 \quad (3)$$

The third term in the right hand side of the Eq. (3) is important in low Reynolds number and its effect vanish by increasing of the Reynolds number and reduction of the drag coefficient. Eq. (3) is in very good agreement with results of the reference [8] and correlates the data with average error of 8.4883%. The maximum difference between values of Eqs. (2) and (3), and experimental data, is related to the transient region where the Reynolds number is in the range of $5.8 \times 10^{-2} \leq Re \leq 4.5$. Fig. 1 demonstrates the variation C_D versus Re for rolling spherical particles.

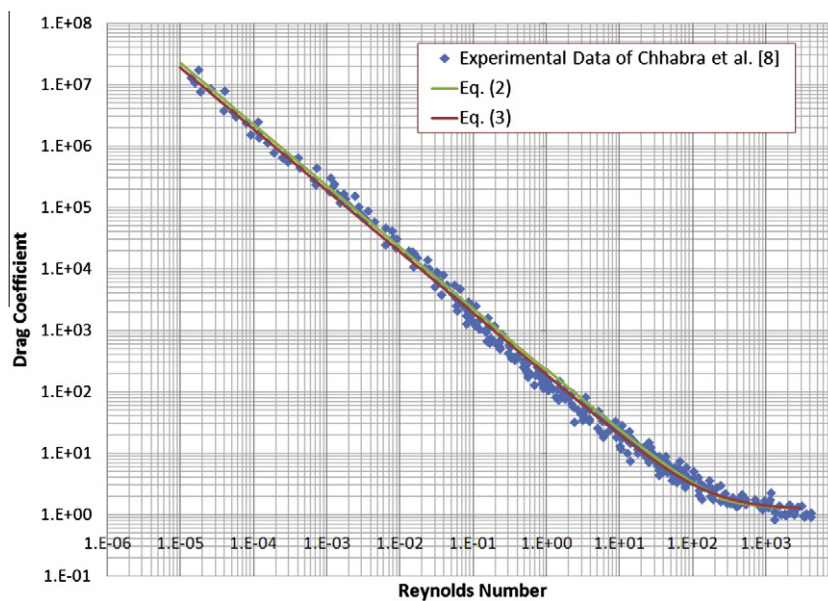


Fig. 1. C_D versus Re for rolling spheres.

Aside from mentioned work of Chhabra et al. [8], all other surveys of the rolling motion of the particles are related to open channels [9–12]. In reality, when a sphere is rolling in a tube, the wall exerts an extra retardation effect on it due to upward motion of the fluid through the eccentric annular gap between the particle and the wall this issue distinguishes the mechanism of rolling in tubes from open channels.

It is clear that a few studies are performed on rolling motion of particles, especially in tubes while it is an important practical issue both in nature and industry. Moreover, most of the previous studies in particles motion and sedimentations are experimental or numerical. However, an exact analytical expression is more opportune for engineering calculations, and is also the evident starting point for a better understanding of the relationship between the physical properties of the sphere-fluid combination and the accelerated motion of the sphere. In addition, In contrast to steady-state motion of particles much less has been reported about the acceleration motion of spherical particles in incompressible Newtonian fluids. The accelerated motion is relevant to many processes such as particle classification, centrifugal and gravity particle collection and/or separation, where it is often necessary to determine the trajectories of particles accelerating in a fluid [13]. Furthermore, for other particular situations, like viscosity measurement using the falling-ball method or rain-drop terminal velocity measurement it is necessary to know the time and distance required for particles to reach their terminal velocities.

This paper aims to present an analytical investigation on time dependent motion of rolling spherical particle in a tube and express an exact solution for rolling particle equation of motion when the Eq. (3) is used as drag coefficient relation for a wide range of Reynolds number.

2. Force analysis and equation of motion

Consider a spherical particle of diameter d and density of ρ_p rolling down a smooth tube having angle of inclination α with the horizontal, and filled with an incompressible Newtonian fluid of density ρ_c . Let u represent the velocity of the sphere at any instant t and g the acceleration due to gravity. Fig. 2 illustrate a schematic view of the present problem.

The forces acting on the spheres are the fluid-drag, F_D , fluid lift force, F_L , buoyancy force, F_B , gravitational force, F_g , solid–solid resistance force due to rolling, F_R , virtual mass force, F_{VM} due to relative acceleration of the fluid around the particle. It is reasonable to postulate that since smooth walled tubes have been studied, F_R is expected to be negligible (or alternately $F_D + F_R$ can be regarded as the total resistance to sphere motion, which is included

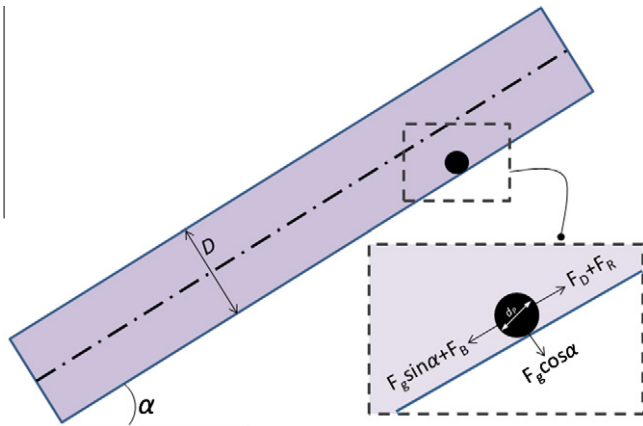


Fig. 2. Exerted forces on a sphere rolling down a tube.

in the drag coefficient). For a sphere with density much larger than that of the fluid, the Basset contribution can be neglected in the force balance equation [11,13–15]. Under these conditions and in the absence of significant effects arising from the lift force [5–8,11], the motion of a sphere rolling down a smooth solid boundary can be described by the equation:

$$\frac{\pi d_p^3}{6} \rho_p \left(1.4 + 2.0 \frac{\rho_c}{\rho_p} \right) \frac{du}{dt} = \frac{\pi d_p^3}{6} \rho_p g \left(1 - \frac{\rho_c}{\rho_p} \right) \sin \alpha - \frac{\pi d_p^2}{8} C_D \rho_c u^2 \quad (4)$$

Substituting drag coefficient equation, Eq. (3) into equation above the unsteady equation of motion of rolling particle is obtained as follows:

$$\frac{\pi d_p^3}{6} \rho_p \left(1.4 + 2.0 \frac{\rho_c}{\rho_p} \right) \frac{du}{dt} = \frac{\pi d_p^3}{6} \rho_p g \left(1 - \frac{\rho_c}{\rho_p} \right) \sin \alpha - \frac{\pi d_p^2}{8} \left[1.2 + \left(\frac{190}{Re} \right) + \left(\frac{1.003 \times 10^{-7}}{Re^2} \right) \right] \rho_c u^2 \quad (5)$$

The Reynolds number for a spherical particle is defined as:

$$Re = \frac{\rho_c u d_p}{\mu_c} \quad (6)$$

By inserting the Reynolds number which is related to the instantaneous velocity of the particle into the equation of motion, Eq. (5), we have:

$$\frac{\pi d_p^3}{6} \rho_p \left(1.4 + 2.0 \frac{\rho_c}{\rho_p} \right) \frac{du}{dt} = \frac{\pi d_p^3}{6} \rho_p g \left(1 - \frac{\rho_c}{\rho_p} \right) \sin \alpha - \frac{\pi d_p^2}{8} \left[1.2 + \left(\frac{190}{\frac{\rho_c u d_p}{\mu_c}} \right) + \left(\frac{1.003 \times 10^{-7}}{\left(\frac{\rho_c u d_p}{\mu_c} \right)^2} \right) \right] \rho_c u^2 \quad (7)$$

Rearranging Eq. (7) based on particle velocity, u , consequently we have:

$$\frac{\pi d_p^3}{6} \rho_p \left(1.4 + 2.0 \frac{\rho_c}{\rho_p} \right) \frac{du}{dt} = \frac{\pi d_p^3}{6} \rho_p g \left(1 - \frac{\rho_c}{\rho_p} \right) \sin \alpha - \frac{15 \pi d_p^2 \rho_c}{100} u^2 - 23.75 \pi d_p \mu_c u - \frac{1.003 \times 10^{-7} \pi \mu_c^2}{8 \rho_c} \quad (8)$$

To simplify the post processes of the problem we had generated four coefficients in the equation above. Therefore, Eq. (8) is reduced to:

$$a \frac{du}{dt} + bu + cu^2 - d = 0 \quad (9)$$

where:

$$a = \frac{\pi d_p^3}{6} \rho_p \left(1.4 + 2.0 \frac{\rho_c}{\rho_p} \right) \quad (10)$$

$$b = (23.75 \pi d_p \mu_c \rho_c) \quad (11)$$

$$c = \left(\frac{15 \pi d_p^2 \rho_c}{100} \right) \quad (12)$$

$$d = \left\{ \frac{\pi d_p^3}{6} \rho_p g \left(1 - \frac{\rho_c}{\rho_p} \right) \sin \alpha - \frac{1.003 \times 10^{-7} \pi \mu_c^2}{8 \rho_c} \right\} \quad (13)$$

In all solutions, it was assumed that particle is begun to roll from the rest where $u(0) = 0$. The main difficulty in solution of Eq. (8) is depended on the non-linear terms due to non-linearity nature of

the drag coefficient. In fact, Eq. (8) is a non-linear ordinary differential equation which could be solved by numerical techniques such as Runge–Kutta method. Here we employed an analytical method calling homotopy perturbation method to define an exact expression for non-linear equation of motion of the rolling spherical particle. We compared our results with the numerical solution of 4th order Runge–Kutta method.

3. Analytical method

As it was shown the current problem was modeled by a non-linear ordinary differential equation. In most situations, analytical solutions cannot be carried out. So, the equation be supposed to be solved using specific methods. In recent decay, much consideration has been dedicated to the newly advanced approaches to get an exact solution of non-linear equations; such as homotopy perturbation method (HPM) [16–41].

The homotopy perturbation method (HPM), formed by He [25–34], is one of the well-known methods to solve various non-linear equations. This method has been used by many authors in [16–24,35–41] and the references therein to handle a wide variety of scientific and engineering requests: linear and non-linear, and homogeneous and inhomogeneous as well. It was shown that this method provides ameliorations over existing techniques. Recently, Jalaal et al. [11–13] used current method for two practical problems include a spherical particle equation of motion falling or rolling in incompressible viscous media. They found an exact expression for particle's velocity during its falling procedure, which is related to sphere-fluid physical properties. They had shown that current analytical method expresses a very simple procedure to gain an accurate developed equation for particle equation of motion. Therefore, in the present work we study the non-linear equation of motion for a single rolling particle and try to obtain its exact solution using the HPM.

The homotopy perturbation method is a combination of the classical perturbation and homotopy techniques. Unlike classical perturbation method, HPM does not need any small parameter. To explain the fundamental initiative of the HPM for solving non-linear differential equations we consider the general form of following non-linear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \quad (14)$$

With the boundary conditions of:

$$B(u, \partial u / \partial n) = 0, \quad r \in \Gamma \quad (15)$$

Table 1
Physical properties of used materials.

Continues phase	Density	Viscosity
Water	996.51	0.001
Ethylene-glycol	1111.40	0.0157
Glycerin	1259.90	0.779

Table 2
Coefficients for different solid–liquid combinations.

Continues phase	Particle diameter, d_p (mm)	Inclination angle, α (°)	a	b	c	d
Water	1	5	0.00003024123088	0.000074610625	0.0004695804248	7.638667428 e-7
		15	0.00003024123088	0.000074610625	0.0004695804248	0.000002268242935
		22	0.00003024123088	0.000074610625	0.0004695804248	0.000003283167712
		30	0.00003024123088	0.000074610625	0.0004695804248	0.000004382231328
	2	15	0.00002419298470	0.000149221250	0.001878321699	0.00001814594348
Ethylene-glycol	3	15	0.00008165132337	0.000223831875	0.004226223822	0.00006124255925
	1	22	0.000003144432067	0.001171386812	0.0005237194650	0.000003061997753
	1	22	0.000003299936317	0.05961388938	0.0005936963775	0.000002776106643

The operator A can be divided into two elements: a linear part L and a non-linear part, N . Therefore, Eq. (14) can be revised as follows:

$$L(u) + N(u) - f(r) = 0 \quad (16)$$

In case that the non-linear Eq. (16) has no small parameter, we can make the following homotopy:

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \quad (17)$$

where,

$$v(r, p) : \Omega \times [0, 1] \rightarrow R \quad (18)$$

In Eq. (18), $p \in [0, 1]$ is an artificial embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of considered equation can be obtained as a power series in p , as following:

$$v = \sum_i^n p^i v_i = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + p^4 v_4 + \dots \quad (19)$$

and the best approximation for the solution is:

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + v_3 + v_4 + \dots \quad (20)$$

When, Eq. (17) correspond to Eqs. (14) and (20) becomes the approximate solution of the considered non-linear equation. By increasing the number of series' terms in Eq. (19) the accuracy of the result will be augmented and result could be used as an exact

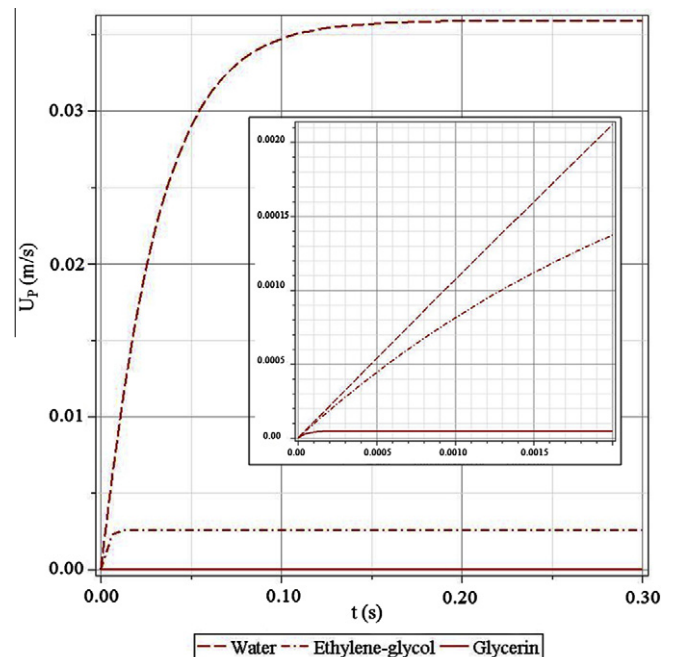


Fig. 3. Instantaneous velocity of a 1 mm diameter sphere rolling in different liquids.

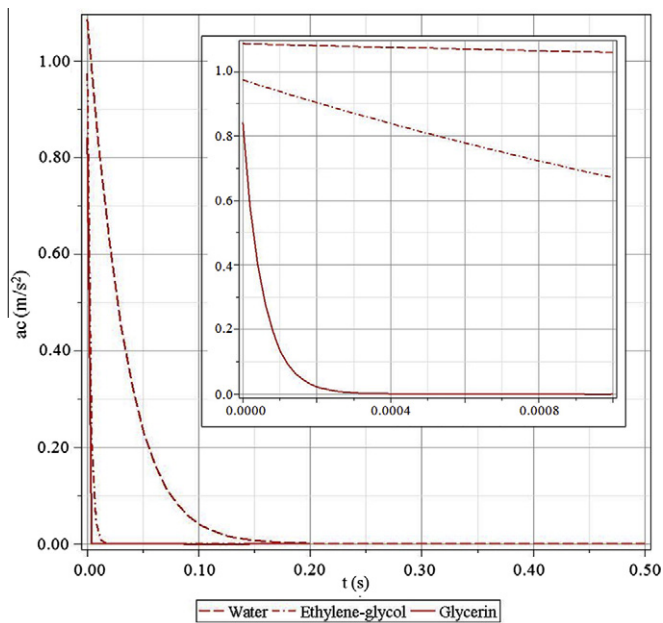


Fig. 4. Acceleration variations of a 1 mm diameter sphere rolling in different liquids.

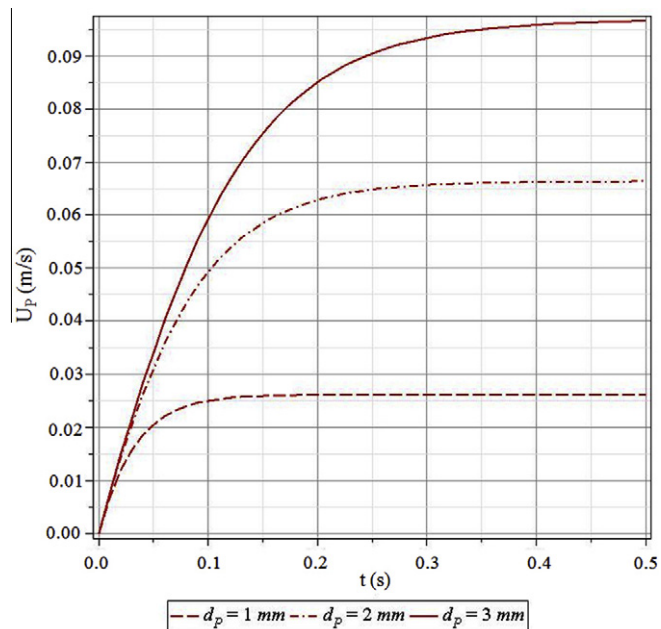


Fig. 6. Instantaneous velocity of three spherical particles rolling in a tube filled with the water.

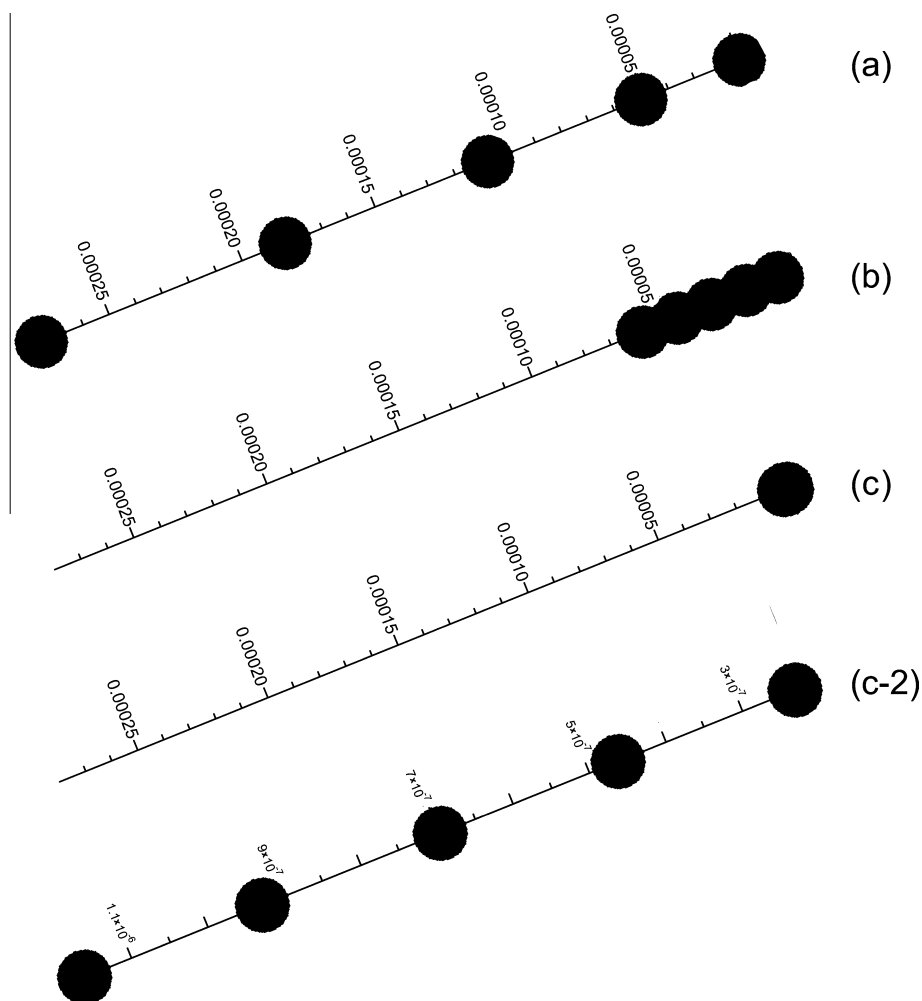


Fig. 5. Position of the rolling particle in acceleration motion (dimensions are in meter, time step is 0.005 s and $\alpha = 22^\circ$). (a) Water, (b) ethylene-glycol, (c) glycerin and (c-2) magnified view of (c).

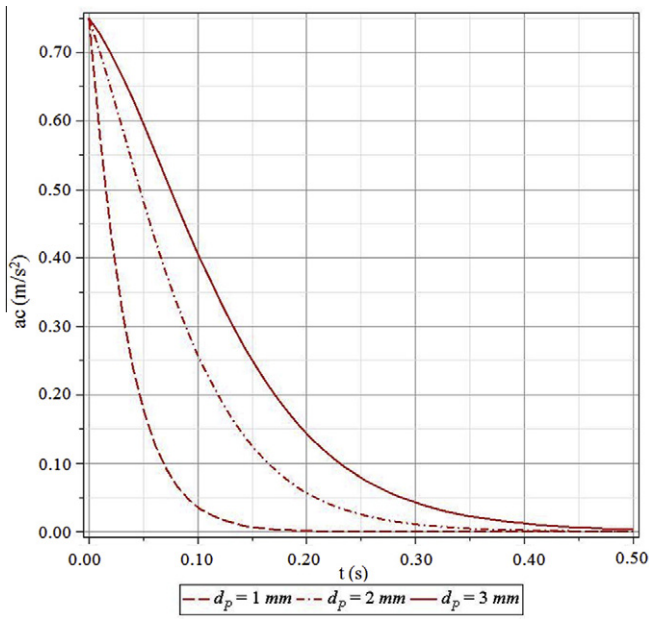


Fig. 7. Acceleration variation versus time for different size of rolling particles.

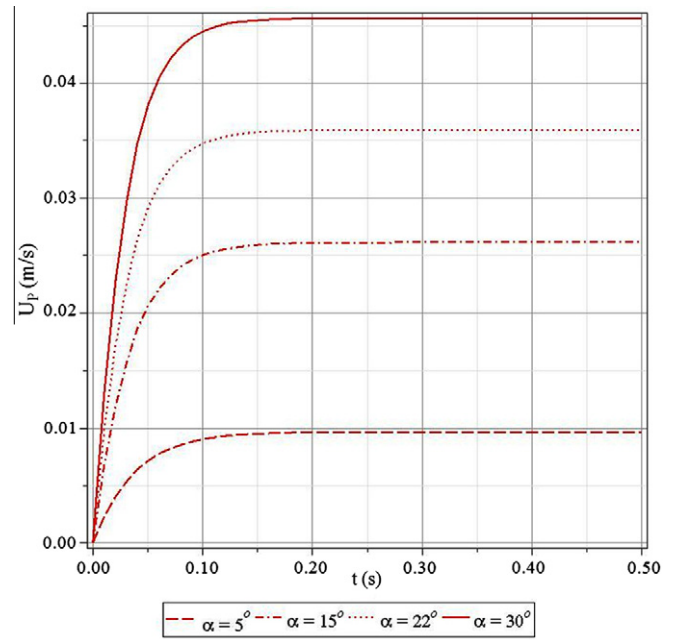


Fig. 9. Velocity variation of a spherical particle rolling in a tube for different inclination angle.

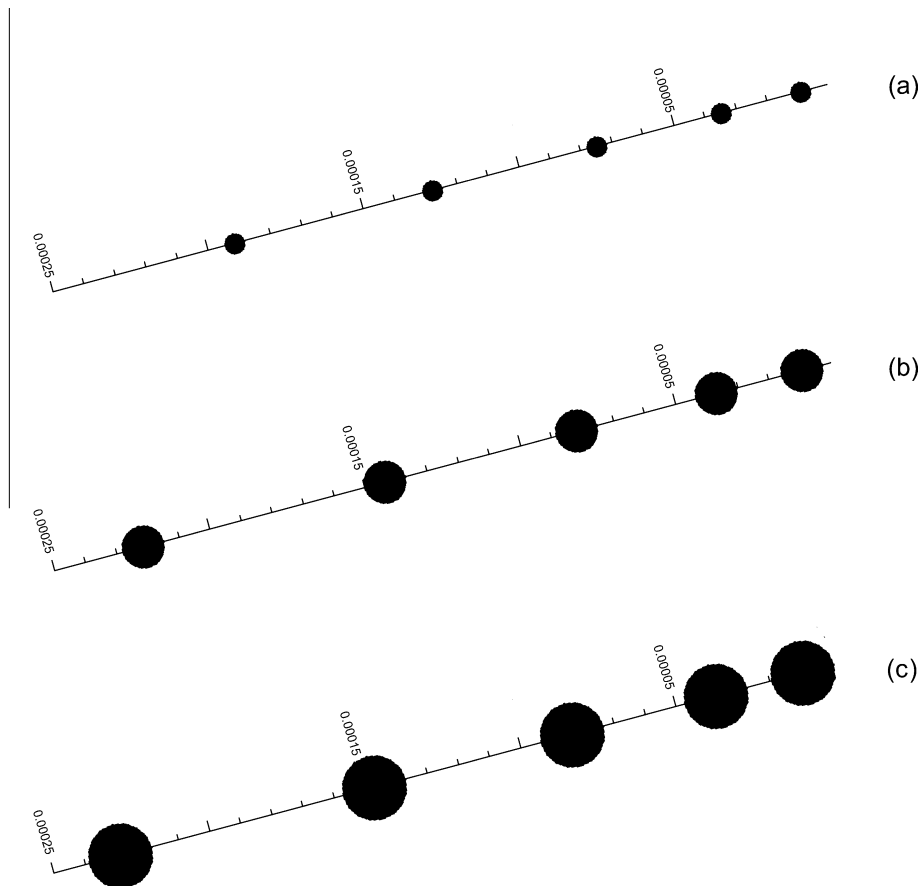


Fig. 8. Displacements of the particles with different diameters rolling down a tube filled with water and inclination angle of 15° (dimensions are in meter and time step = 0.005 s). (a) $d_p = 1$ mm, (b) $d_p = 2$ mm and (c) $d_p = 3$ mm.

solution. The number of series' terms using in the solution is dependent on nature of the problem and order of the non-linearity. Convergence and stability of this scheme are publicized in [22].

4. Results and discussion

Considering Eq. (9), according to the HPM, we can construct the homotopy as follow:

$$H(v, p) = a \frac{dv}{dt} + bv - \left(a \frac{du_0}{dt} + bu_0 \right) + p \left(a \frac{du_0}{dt} + bu_0 \right) + p(c v^2 - d) = 0 \quad (21)$$

where two first terms in the LHS of the Eq. (9) are assumed as the linear part of the equation. Substituting v series from Eq. (19) into Eq. (21) and some simplification and rearranging based on powers of p -terms, we have:

$$p^0 : a \frac{dv_0}{dt} + bv_0 = 0 \quad t = 0, \quad v_0 = 0 \quad (22)$$

$$p^1 : a \frac{dv_1}{dt} + bv_1 + cv_0^2 - d = 0 \quad t = 0, \quad v_1 = 0 \quad (23)$$

$$p^2 : a \frac{dv_2}{dt} + bv_2 + 2cv_0v_1 = 0 \quad t = 0, \quad v_2 = 0 \quad (24)$$

$$p^3 : a \frac{dv_3}{dt} + bv_3 + cv_1^2 + 2cv_0v_1 = 0 \quad t = 0, \quad v_3 = 0 \quad (25)$$

$$p^4 : a \frac{dv_4}{dt} + bv_4 + 2cv_1v_2 + 2cv_0v_3 = 0 \quad t = 0, \quad v_4 = 0 \quad (26)$$

$$p^5 : a \frac{dv_5}{dt} + bv_5 + cv_2^2 + 2cv_1v_3 + 2cv_0v_4 = 0 \quad t = 0, \quad v_5 = 0 \quad (27)$$

$$p^6 : a \frac{dv_6}{dt} + 2cv_0v_5 + 2cv_2v_3 + 2cv_1v_4 = 0 \quad t = 0, \quad v_6 = 0 \quad (28)$$

As it is obvious in equations above, unlike traditional perturbation method, Unlike traditional perturbation methods, we keep $v_0(0) = v(0)$, and $\sum_{i=1}^{\infty} v_i(0) = 0$. By continuing above terms, higher accuracy will be gained. Solving Eqs. (22)–(28) considering appropriate initial conditions, we have:

$$v_0 = 0 \quad (29)$$

$$v_1 = \frac{d}{b} - \frac{de^{-\frac{bt}{a}}}{b} \quad (30)$$

$$v_2 = 0 \quad (31)$$

$$v_3 = -\frac{cd^2 \left(\left(ae^{\frac{bt}{a}}/b \right) - 2t - \left(ae^{-\frac{bt}{a}}/b \right) \right)}{b^2 a} \quad (32)$$

$$v_4 = 0 \quad (33)$$

$$v_5 = \frac{c^2 d^3 e^{-\frac{bt}{a}}}{b^5} - \frac{1}{a^2 b^5} \left(c^2 d^3 \left(-2a^2 e^{\frac{bt}{a}} + 2b^2 t^2 + 2a^2 e^{-\frac{bt}{a}} + 2abt + 4ae^{\frac{bt}{a}}bt + a^2 e^{-\frac{2bt}{a}} \right) e^{\frac{bt}{a}} \right) \quad (34)$$

$$v_6 = 0 \quad (35)$$

As it is apparent, solution of terms vary periodically and in each step more duration of the particle motion is covered. We continued Eq. (19) up to 15th power of p to gain an exact solution for unsteady particle movement.

The final expression for particle equation of motion can be found as follows:

$$u = \sum_{i=0}^{13} \lim_{p \rightarrow 1} p^i v_i$$

$$= \frac{d}{b} - \frac{de^{\frac{bt}{a}}}{b} - \frac{cd^2 \left(\left(ae^{\frac{bt}{a}}/b \right) - 2t - \left(ae^{-\frac{bt}{a}}/b \right) \right)}{b^2 a} + \frac{c^2 d^3 e^{\frac{bt}{a}}}{b^5}$$

$$- \frac{1}{a^2 b^5} \left(c^2 d^3 \left(-2a^2 e^{\frac{bt}{a}} + 2b^2 t^2 + 2a^2 e^{\frac{bt}{a}} + 2abt + 4ae^{\frac{bt}{a}}bt + a^2 e^{\frac{2bt}{a}} \right) e^{\frac{bt}{a}} \right)$$

$$- \frac{4c^3 d^4}{b^7 e^{\frac{bt}{a}}} + \frac{2c^3 d^4 t}{b^6 a^2 e^{\frac{bt}{a}}} + \frac{4c^3 d^4}{b^7 \left(e^{\frac{bt}{a}} \right)^2} - \frac{5c^3 d^4}{b^7} + \frac{4c^3 d^4 t^3}{3b^4 a^3 e^{\frac{bt}{a}}} + \frac{8c^3 d^4 t^2}{b^5 a^2 \left(e^{\frac{bt}{a}} \right)^2} + \frac{12c^3 d^4 t}{b^6 a \left(e^{\frac{bt}{a}} \right)^2}$$

$$+ \frac{4c^3 d^4}{b^7 \left(e^{\frac{bt}{a}} \right)^3} + \frac{4c^3 d^4 t^2}{b^5 a^2 e^{\frac{bt}{a}}} + c + \frac{c^3 d^4}{b^7 \left(e^{\frac{bt}{a}} \right)^4} + \frac{14c^4 d^5}{e^{\frac{bt}{a}} b^9} - \frac{8c^4 d^5 t}{a \left(e^{\frac{bt}{a}} \right) b^8} \quad (36)$$

$$- \frac{6c^4 d^5}{b^9 \left(e^{\frac{bt}{a}} \right)^4} - \frac{c^4 d^5}{b^9 \left(e^{\frac{bt}{a}} \right)^5} - \frac{18c^4 d^5 t^2}{a^2 b^7 \left(e^{\frac{bt}{a}} \right)^3} - \frac{30c^4 d^5 t}{ab^8 \left(e^{\frac{bt}{a}} \right)^3} - \frac{8c^4 d^5}{b^9 \left(e^{\frac{bt}{a}} \right)^2}$$

$$+ \frac{14c^4 d^5}{b^9} - \frac{32c^4 d^5 t}{ac^8 \left(e^{\frac{bt}{a}} \right)^2} - \frac{8c^4 d^5 t^2}{a^2 b^7 e^{\frac{bt}{a}}} - \frac{4c^4 d^5 t^3}{a^3 b^6 e^{\frac{bt}{a}}} - \frac{32c^4 d^5 t^3}{3a^3 b^6 \left(e^{\frac{bt}{a}} \right)^2}$$

$$- \frac{32c^4 d^5 t^3}{a^2 b^7 \left(e^{\frac{bt}{a}} \right)^2} - \frac{2c^4 d^5 t^4}{3a^4 b^5 e^{\frac{bt}{a}}}$$

Here we used more than 10 p -terms for a reliable result. There are many works which show that HPM is able to express an accurate result where only one or two iterations are enough [38–41]. The convergence and validation of current analytical solution are demonstrated in [11,13] and is shortly described in Appendix 1. Current expression can be use easily by calculating the coefficients. Three liquids with different densities and viscosities are selected as continues phases. Properties of the fluids are listed in the Table 1.

In the modeling, Aluminum with density of $\rho_s = 2702.00 \text{ kg/m}^3$ is used for dispersed phase (particle). Current sphere-fluid combinations provide a wide range of Reynolds number where in high and low viscosity liquid (glycerin and water) a creeping and high-acceleration motion is expected, respectively.

By substituting the values mentioned in the Table 1 into Eqs. (10)–(13) the coefficients needed for the solution will be gained. Effects of continues phase viscosity (while the difference between liquid densities is relatively negligible), inclination angle of the tube and the diameter (size) of the particle are investigated with the coefficients listed in the Table 2.

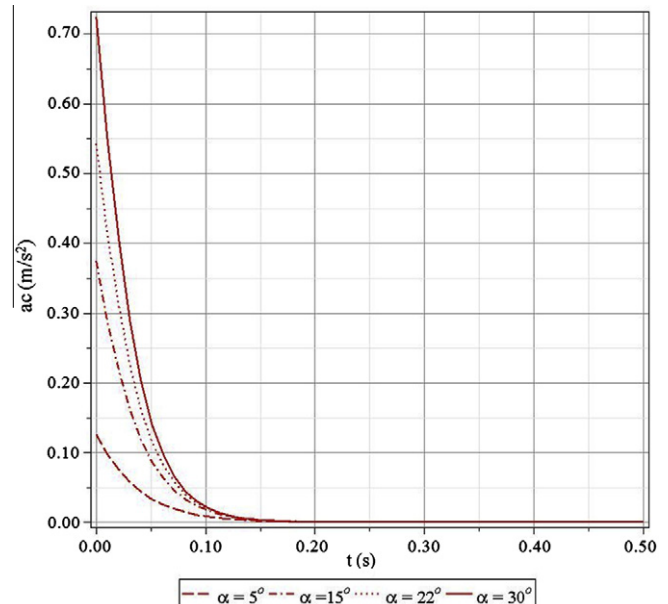


Fig. 10. Instantaneous acceleration for different inclination angle (the tube is filled with water and the particle diameter is 1 mm).

Fig. 3 depicts the variation of velocity versus time for a spherical particle of 1 mm diameter rolling in a tube with inclination angle of 22° where the initial time of the rolling is magnified. A particle rolling down a tube in a fluid under the influence of gravity will accelerate until the gravitational force is balanced by the resistance forces that include buoyancy and drag. The constant velocity reached at that stage is called the “terminal velocity” or “settling velocity”. Knowledge of the terminal velocity of solids falling or rolling in liquids is required in many industrial applications.

Clearly, the value of the rolling terminal velocity is increased by reduction of the viscosity. The acceleration region is very short for glycerin because of high hydrodynamic resistive force and the particle reaches the terminal velocity rapidly. Fig. 4 clearly show the variation of the acceleration during rolling procedure for same conditions of curves in the Fig. 3.

Fig. 4 demonstrates that the acceleration duration for a sphere rolling in a tube is very short for a high viscose liquid such as glycerin. On the other hand, this time increased with decreasing of the

liquid phase viscosity and should be considered. The position of the particle which is obtained by a simple integration of Eq. (36) is demonstrated in Fig. 5 for different considered fluids. The inclination angle is fixed at 22° and the time step (time difference for each integration) is 0.005 s.

Fig. 5 noticeably explains the acceleration motion of the particles when the distance is increased in each time step. Fig. 5c, magnified view in (c-2), show the creeping flow of the particle due to high viscosity of the glycerin.

The influence of particle size is studied where the diameter of the particles is varied in the range of $1\text{ mm} < d_p < 3\text{ mm}$. Fig. 6 shows the variations of the velocity for three different particles rolling in a tube inclined with the angle of 15° and filled with the water.

Fig. 6 evidently shows that the value of the terminal velocity in a rolling procedure is significantly increased with adding to the particle size. In addition, the acceleration duration is significantly

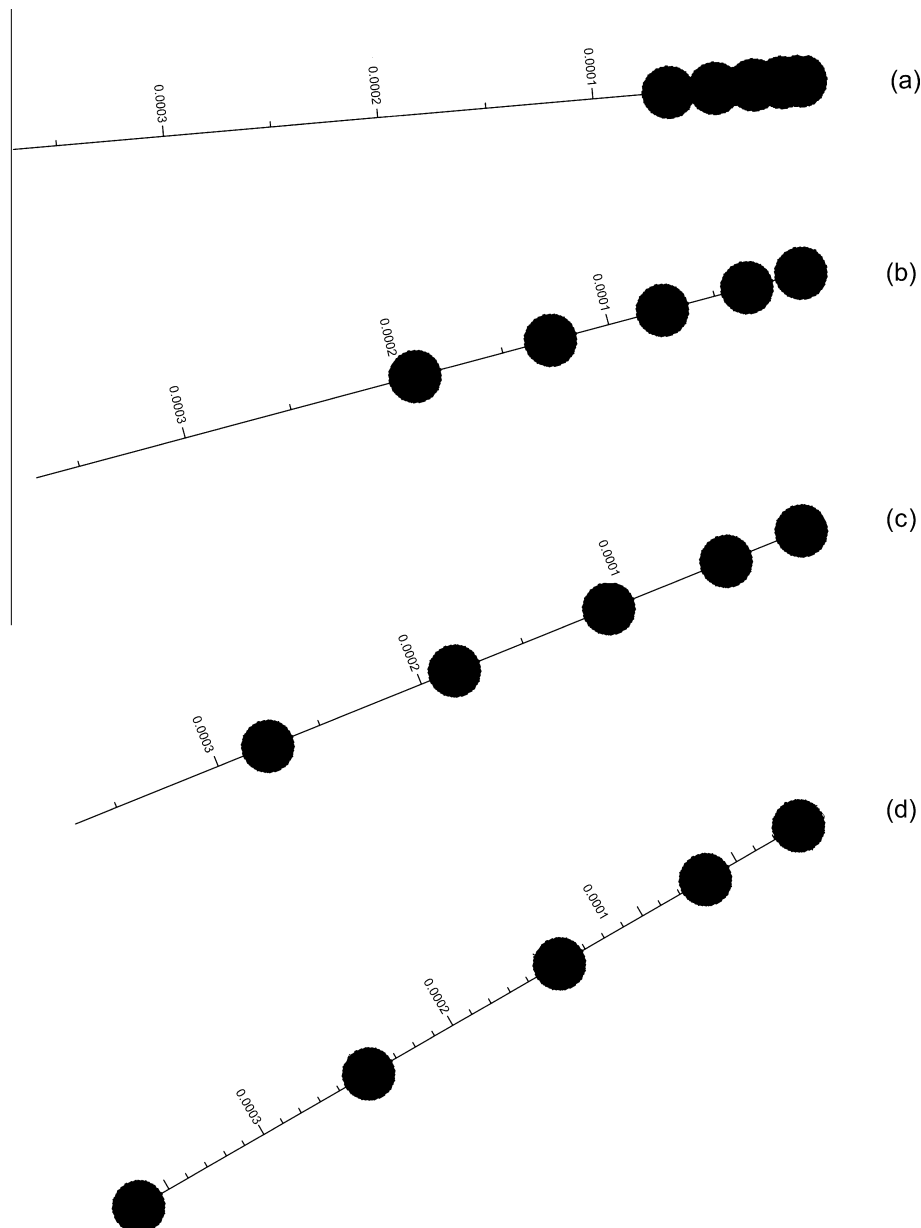


Fig. 11. Different position of the rolling spherical particle of 1 mm diameter in a tube filled with water (dimensions in meter and time step = 0.005). (a) $\alpha = 5^\circ$, (b) $\alpha = 15^\circ$, (c) $\alpha = 22^\circ$ and (d) $\alpha = 30^\circ$.

longer for larger particles. The effect of particle size on rolling acceleration is illustrated in Fig. 7.

As it is depicted in the Fig. 7, the acceleration time is highly depended on the particle size. For instance, the acceleration duration is about 0.4 s for a particle of 2 mm diameter while this value is less than 0.2 s for a particle with a diameter of 1 mm. For particle of 3 mm a different manner of acceleration in the initial period of the motion is seen. This fact is due to effect of the gravitational force which increased by adding to volume (mass) of the particle. To depict the effect of particle size on acceleration rolling motion, the positions of the particles of different size are illustrated in Fig. 8 when the fluid is water and the angle of inclination is fixed in 15°.

An important parameter in rolling motion of the particles is the inclination of the tube. The effect of inclination angle is appeared in the gravitational force term in the equation of motion and its value increases with adding to the inclination angle. In the current work, the inclination angle is changed from 5° to 30° due to limitation of the used drag coefficient obtained from Chhabra et al. experimental points [8]. The higher inclination angles need their specific empirical investigation and new drag correlation. Fig. 9 shows the effect of the tube inclination angle on instantaneous velocity of a particle with a diameter of 1 mm rolling in a tube filled with the water.

Clearly, the terminal velocity is larger for higher inclination angles. The main point is the insignificant effect of the inclination on acceleration duration. To study this fact in details, the alteration of the acceleration is illustrated in the Fig. 10 for four different angles of inclination.

The fascinating point is that the particle reaches the zero acceleration (terminal velocity) in a same time for any angle of inclination. It means inclination has not a noteworthy influence on the acceleration region of a rolling particle. The interesting behavior of rolling particles in a tube for different angle of inclinations is illustrated in Fig. 11.

It is obvious that Eq. (8) gained by HPM easily demonstrate the physical behavior of a rolling particle when the displacement increases with adding to angle of inclination.

5. Conclusion

Many problems in aerosol, two-phase gas–solid and solid–liquid flows include particle equation of motion. Most of the pervious works in this field are related to numerical works due to non-linearity of the equation. Besides, most of the studies in the literature investigate the freely falling particles. In the present work, the unsteady rolling motion of a spherical particle in a tube was surveyed analytically using homotopy perturbation method. An exact expression is gained for particle velocity in the acceleration motion. Solution was performed for a general form of equation and extended for some practical conditions. The effects of continuous phase viscosity, particle size and angle of inclination were studied. Results clearly demonstrate the hydrodynamics behavior of the particle rolling in a tube in different conditions. Moreover as an important result, it was understood that inclination of the tube does not affect the acceleration duration while other physical properties significantly affects this parameter. Current results show the capability of the HPM as an analytical method and can be used in different related analytical and numerical works.

Appendix 1

As it is obvious, in Eq. (36), solutions of terms vary periodically and in each step more duration of particle motion is covered. As an example, Eq. (9) was solved with following constants:

$$a = b = c = d = 1 \quad (\text{Ap.1})$$

therefore, Eq. (9) could be rewritten as follow:

$$\frac{du}{dt} + u + u^2 - 1 = 0, \quad u(0) = 0 \quad (\text{Ap.2})$$

Eq. (Ap. 2) was solved for different steps of Eq. (19) and results were compared with Runge–Kutta solution. Fig. Ap1 demonstrates the velocity variation with time for different condition of solution.

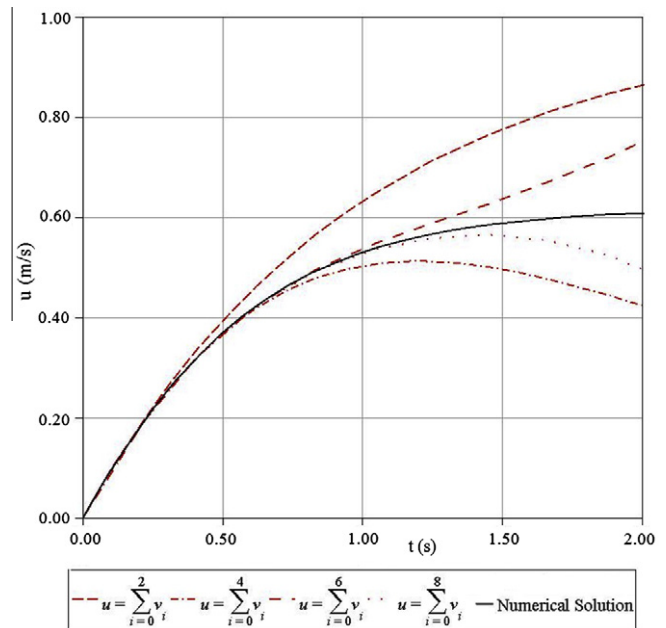


Fig. Ap1. HPM and numerical solutions of Eq. (Ap. 2).

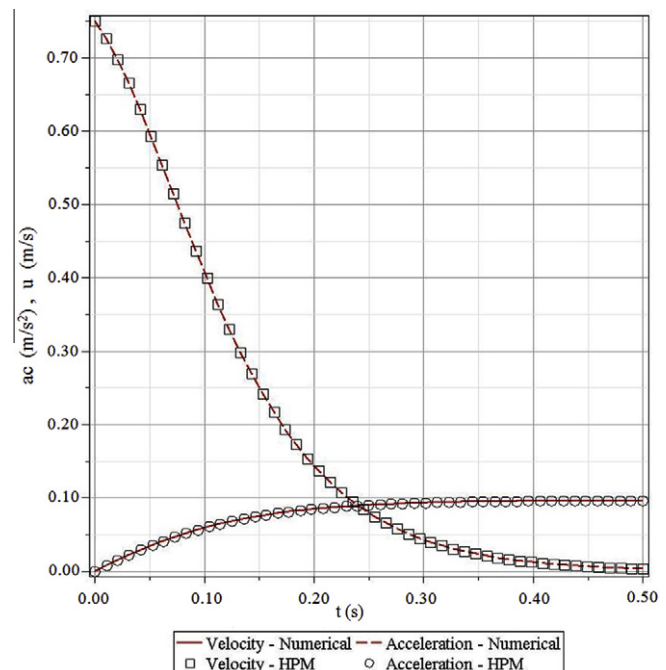


Fig. Ap2. HPM and numerical solutions of Eq. (8) when the fluid is water, $\alpha = 15^\circ$ and $d_p = 3$ mm.

As it was mentioned and illustrated in figure above, by increasing of series' terms, the accuracy of HPM solution is enhanced and larger period of acceleration motion of the particle is covered. From mathematical point of view, by adding to series terms in the homotopy solution, the series is become closer to the exact solution. In the current study, we augmented the terms until the terminal velocity of particle is reached. HPM and numerical solutions of a real sphere-liquid combination are compared in Fig. Ap2 for a particle of 3 mm diameter rolling in a tube with inclination angle of 15° and filled with water.

The analytical expression obtained using HPM is completely validated with the numerical solution and can be used for a wide range of applications related to aerosols and powder technologies.

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